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Reply by Author to J. T. Ohrenberger and C. B. Cohen

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In Ref. 1 the author proposed a new method for the integration of the Karman momentum integral equation for compressible flow with arbitrary streamwise pressure gradient. In the same issue of the AIAA Journal, Ohrenberger and Cohen[2] made several comments (mostly unfavorable) with regard to this method. It is the intent of this note to reply to these comments.

1) A comparison is made between my expression for the local momentum thickness (squared), which is obtained from an integration of the Karman momentum integral equation [Eq. (14) of Ref. 1], and that obtained from local similarity exclusively [Eq. (2) of Ref. 2]. They differ by the ratio $[\theta_{tr}(x)/\theta_{tr}(0)]^2$ which appears in the Ohrenberger-Cohen result. "This ratio," they say, "can be interpreted as a correction factor which partially accounts for the effects of the variation of β, $T_w/T_{e\text{stag}}$ and M_e along the body. The fact that this correction factor is not present" in Eq. (14) of Ref. 1 "is what appears to be unsatisfactory . . . "

In reply, it is readily shown that the variation of β and M_e along the body is accounted for in my expression for θ^2. (The wall temperature and hence the ratio $T_w/T_{e\text{stag}}$ were considered constant in my analysis.) From a combination of Eqs. (4) and (5) of Ref. 1, β can be written

$$\beta = 2\frac{T_{e\text{stag}}}{T_e}\frac{du_e}{dx}\cdot\left(\int_0^x p_e^\alpha u_e r_w^{2\epsilon}dx \Big/ p_e^\alpha u_e^2 r_w^{2\epsilon}\right) \quad (1)$$

Substituting this expression into Eq. (14), Ref. 1, gives

$$\theta^2 = (1+\epsilon)\theta_{\text{stag}}^2\frac{(du_e/dx)_{\text{stag}}}{du_e/dx}\times \beta\left(1+\frac{\gamma-1}{2}M_e^2\right)^{(3-2\gamma)/(\gamma-1)} \quad (2)$$

so that θ^2 does depend on the variation of β and M_e along the body.

2) Ohrenberger and Cohen question the validity of my result by citing the example of the supersonic flow over a blunted wedge with a very small nose radius relative to the base height. They say that in the limit of zero nose radius, Eq. (14) of Ref. 1 does not reduce to the flat-plate case. It does not, but that is because Eq. (14) is for blunt-nosed bodies and should not reduce to the flat-plate case. In the limit of zero nose radius, i.e., for sharp-nosed bodies, it is necessary to return to Eq. (13) of Ref. 1 and to reevaluate the quantities at $x = 0$. When this is done, Eq. (13), Ref. 1, becomes

$$\theta^2 = (1+2\epsilon+m_0)\left(\theta^2\frac{u_e}{x}\right)_0\times \left(\frac{1+[(\gamma-1)/2]M_e^2}{1+[(\gamma-1)/2]M_{e_0}^2}\right)^{(2-\gamma)/(\gamma-1)}\cdot \left(\int_0^x p_e^\alpha u_e r_w^{2\epsilon}dx \Big/ p_e^\alpha u_e^2 r_w^{2\epsilon}\right) \quad (3)$$

where the contents of the first two brackets on the right-hand side can be written

$$(1+2\epsilon+m_0)\left(\theta^2\frac{u_e}{x}\right)_0 = 2\nu_{w_0}\left(\frac{T_e}{T_w}\right)_0^2\times \left\{\left[f''(0)-\beta\lim_{\eta\to\infty}(\eta-f)-\beta\int_0^\infty S d\eta\right]\Big/\left(1+\beta\right)\right\}^2_{x=0} \quad (4)$$

Quantities with the subscript $(\ )_0$ refer to conditions at $x = 0$; m_0 is the exponent in the expression for the velocity at the outer edge of the boundary layer at $x = 0$, namely, $u_{e_0} = a_1x^{m_0}$. Equations (3) and (4) are general and hold for sharp-nosed and blunt-nosed bodies. For the latter, $m_0 = 1$ and Eq. (3) reduces immediately to Eq. (14), Ref. 1. For the supersonic flow over a sharp-nosed straight wedge with an attached shock, Eqs. (3) and (4) give the exact flat-plate result. Hence, this objection of Ohrenberger and Cohen to my blunt-body result is not valid.

For the problem posed by Ohrenberger and Cohen—the supersonic flow over a blunt-nosed wedge with a very small nose radius—I would use my Eq. (14) and evaluate the integral in two parts, the first part containing the variable p_e and u_e and the second part (which can be integrated immediately) containing the constant p_e and u_e.

3) Ohrenberger and Cohen engage in a lengthy discussion on the advisability of my using Eq. (12), Ref. 1, for the F function. To justify this choice I will start at the unnumbered equation preceding Eq. (12), Ref. 1, and insert an additional step which was omitted (in the interests of conciseness) in the published note. Starting with the unnumbered equation and using Eqs. (4) and (10) of Ref. 1, we find

$$F_{\text{loc sim}} = \tfrac{1}{2}(d\xi/\xi dx)(u_e\theta^2/\nu_{\text{ref}})_{\text{loc sim}} \quad (5)$$

The subscript, loc. sim., is then dropped, and Eq. (12) of Ref. 1 results. It is the dropping of the subscript loc. sim., which requires justification. My justification is that it is consistent with the procedure followed by previous investigators (Refs. 3 and 4, for example) who have curve-fitted (by straight lines) the results of locally similar solutions to permit integration of the Karman momentum integral equation. To develop this point further, consider the two-dimensional flow of an incompressible fluid with pressure gradient. The Karman momentum integral equation can then be written

$$\frac{d}{dx}\left(\frac{u_e\theta^2}{\nu}\right)-\frac{u_{e_x}}{u_e}\left(\frac{u_e\theta^2}{\nu}\right) = G \quad (6)$$

where

$$G = 2\frac{\tau_w\theta}{\mu u_e}-4\frac{u_{e_x}\theta^2}{\nu}-2\frac{u_{e_x}\theta\delta^*}{\nu} \quad (7)$$

Thwaites[3] approximated the G function by the straight line

$$G(u_{e_x}\theta^2/\nu) = a - b(u_{e_x}/u_e)(u_e\theta^2/\nu) \quad (8)$$

(where a and b are constants) using the results of locally similar solutions for this purpose. The substitution of Eq. (8) into Eq. (6) permits the latter to be integrated, giving the result which is contained, for example, in Ref. 5. Thwaites[3] did not tag the grouping $u_e\theta^2/\nu$ with the subscript, loc. sim., although the G function is obtained from locally similar solutions. I have followed the same procedure; that is, I have

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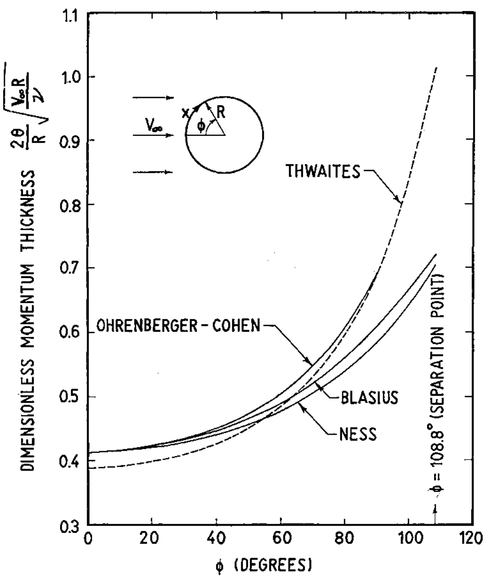

Fig. 1 Comparison with the Blasius series solution of the momentum thickness from several different authors.

dropped the tag on the grouping $u_e\theta^2/\nu_{ref}$ and assumed it to be the dependent variable in the differential equation (2) of Ref. 1.

4) A comparison is made in Fig. 1 of the momentum thickness resulting from several different expressions for the flow of an incompressible fluid around a two-dimensional circular cylinder. The standard for comparison is the Blasius series solution. For this flow, the theoretical potential velocity distribution is $u_e = 2V_\infty \sin\phi$ where V_∞ is the freestream velocity and ϕ is the angle measured from the forward stagnation point; ϕ is related to the distance along the surface x and the cylinder radius R by $\phi = x/R$. For this velocity distribution, the Blasius series solution gives the separation point at $\phi = 108.8°$.

The Blasius curve in Fig. 1 is repeated from Ref. 6; the Ohrenberger-Cohen curve (which is based on a variation of my method) results from the application of Eq. (8), Ref. 2, and is reproduced from Ref. 7; the Thwaites curve is obtained from Ref. 5, whereas the Ness curve is based on Eq. (14) of Ref. 1. For the velocity distribution $u_e = 2V_\infty \sin\phi$, the Thwaites and Ness equations are readily integrated and give, respectively,

$$[(2\theta/R)(V_\infty R/\nu)^{1/2}]_{\text{Thwaites}} = (0.949/\sin^3\phi)[\tfrac{8}{15} - \tfrac{4}{15}(\cos\phi)(\sin^2\phi + 2) - \tfrac{1}{5}\sin^4\phi\cos\phi]^{1/2} \quad (9)$$

$$[(2\theta/R)(V_\infty R/\nu)^{1/2}]_{\text{Ness}} = 0.584/(1 + \cos\phi)^{1/2} \quad (10)$$

The Thwaites result is included to provide a comparison between the method of curve-fitting the results of similar solutions and the new method proposed in Ref. 1. The Thwaites curve does not satisfy the exact value at the stagnation point, whereas the curves obtained by the new method do. (This is a built-in feature in the new method in that the exact value at $x = 0$ is always obtained, regardless of the geometry, for incompressible or compressible flow.) The Thwaites curve and the Ohrenberger-Cohen curve (which they terminated at $\phi = 90°$ in Ref. 7) show wide divergence from the Blasius series solution at high angles of ϕ. The Ness curve, however, over the complete range from the stagnation point to the separation point, does not vary more than 4% (or 5%†) from the Blasius result.

† Because of the small size of Fig. 12.8 of Ref. 6, the coordinates of the Blasius curve are not known as accurately as the coordinates of the other three curves.

It appears, therefore, that the new method gives good results, at least for incompressible flow. The validity of this method for the more general case of a compressible fluid with heat transfer awaits confirmation.

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Commenter's Reply to N. Ness

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We have reviewed the reply presented in this issue by N. Ness, defending his method[1] for determining boundary-layer momentum thickness, and we believe that our original criticisms[2] remain valid and that the suggested modification to his method is required. Specific comments on the four points discussed in his reply follow:

1. Ness, in his reply, has simply recast his original result for θ^2 [Eq. (14) of Ref. 1] in terms of β and M_e. Since this does not basically change the result, the fact still holds that the effects of the variation of β, $T_w/T_{e_{stag}}$, and M_e along the body are only partially accounted for in the Ness method (as demonstrated in Ref. 2). Reference 2 shows that the factor $[\theta_{tr}(x)/\theta_{tr}(0)]$, not included by Ness, can be important. Under the local similarity assumption, this factor also depends on β, $T_w/T_{e_{stag}}$, and M_e.

2. We cited in Ref. 2 the example of the blunted wedge in supersonic flow as indication that Ness' result for the momentum thickness failed by itself to approach the proper limit far back on the blunted wedge. Ness replies that this is because Eq. (14) of Ref. 1 for θ^2 is for blunt-nosed bodies (rather than sharp-nosed bodies), and hence "should not reduce to the flat-plate case." This explanation seems to miss the main point of our argument, which is the fact that bluntness effects far from the stagnation point of a blunt wedge are negligible, and, hence, in that region $\theta(x)$ increases at the same rate as on a flat plate. The conclusion is that Ness' result is in error by the ratio $[\theta_{tr}(x)/\theta_{tr}(0)]$ in this limit.

3. Ness justifies his chosen form for F [Eq. (12) of Ref. 1] by noting that it is correct for the case of a similarity boundary layer, and furthermore that "previous investigators" have taken relations between parameters which are only strictly valid for the similarity case and assumed them to be valid in the nonsimilar case as well (the method of Thwaites, for example). We suggest that the success of this procedure

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